

Assignment 3

Submission Deadline: **13 October, 2026** at 23:59

Course Website: <https://moodle-app2.let.ethz.ch/course/view.php?id=28772>

Exercises

You can get feedback for Exercise 2 by handing in your solution as pdf via Moodle before the deadline.

1. Linear functional (in-class) (★☆☆)

a) Let $n \in \mathbb{N}^+$. Consider the function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$T(\mathbf{x}) = \sum_{k=1}^n kx_k$$

for all $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)^\top \in \mathbb{R}^n$. Prove that T is a linear functional.

b) Let $n \in \mathbb{N}^+$ with $n \geq 2$ be arbitrary. Consider the function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$T(\mathbf{x}) = \sum_{k=1}^n (x_k)^k$$

for all $\mathbf{x} = (x_1 \ x_2 \ \cdots \ x_n)^\top \in \mathbb{R}^n$. Is T a linear functional?

2. Matrix powers (hand-in) (★☆☆)

For a natural number $k \geq 1$, we define the k -th power of a square matrix A as the matrix multiplication

$$A^k = \underbrace{AA \cdots A}_{k \text{ times}}$$

Moreover we define $A^0 = I$, where I is the identity matrix.

Consider the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

Use induction to show that for any $k \geq 0$ the k -th power of the matrix A is

$$A^k = \begin{bmatrix} 1+k & k \\ -k & 1-k \end{bmatrix}$$

3. Reconstruct a linear transformation (★☆☆)

a) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}.$$

Determine the general formula for $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ with $x, y \in \mathbb{R}$.

b) Find a matrix A such that $T_A = T$.

4. Linear transformation (☆☆☆)

Let $m, n \in \mathbb{N}^+$ and consider an arbitrary $m \times (n+1)$ matrix

$$A = \begin{bmatrix} | & | & & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n & \mathbf{v}_{n+1} \\ | & | & & | & | \end{bmatrix}$$

with columns $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in \mathbb{R}^m$. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the function defined by

$$T(\mathbf{x}) = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

for all $\mathbf{x} = (x_1 \ x_2 \ \dots \ x_n)^\top \in \mathbb{R}^n$. Prove that T is a linear transformation if and only if $\mathbf{v}_{n+1} = \mathbf{0}$.

5. A formula for the Fibonacci numbers (★★★★)

The Fibonacci numbers $(F_n)_{n \geq 0}$ are defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Let

$$\varphi = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1 - \sqrt{5}}{2}$$

be the two solutions of $z^2 = z + 1$.

a) Consider the function

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x \end{pmatrix}$$

and define

$$\mathbf{v}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \quad \text{for } n \geq 0.$$

Prove that T is a linear transformation and that $\mathbf{v}_{n+1} = T(\mathbf{v}_n)$ for every $n \geq 0$.

b) For $r \in \{\varphi, \psi\}$, define the linear functional

$$L_r: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad L_r \begin{pmatrix} x \\ y \end{pmatrix} = rx + y.$$

Prove that

$$L_r(T(\mathbf{v})) = rL_r(\mathbf{v})$$

for every $\mathbf{v} \in \mathbb{R}^2$ and use this equation and induction on n to show that

$$rF_{n+1} + F_n = r^{n+1}$$

for every $n \geq 0$ and every $r \in \{\varphi, \psi\}$.

c) Use $rF_{n+1} + F_n = r^{n+1}$ for $r \in \{\varphi, \psi\}$, obtained in part (b), to prove that

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

for every $n \geq 0$.