

Diskrete Mathematik HS2026 — Prof. Dennis HOFHEINZ

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Exercise sheet 4

This is the exercise sheet number 4. The difficulty of the questions and exercises are rated from very easy (★) to hard (★★★★). The graded exercise is Exercise 4.6. If your interview is in the week of the **12/10/2026**, please prepare this exercise for the interview. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 4.1 Pigeon set (★★)

Let n be an integer such that $n \geq 3$, and let S be the set $S = \{1, 2, \dots, n-1\}$.

Prove that every subset of S of size $\lfloor \frac{n}{2} \rfloor + 1$ contains two elements whose sum is n .

The floor function $x \mapsto \lfloor x \rfloor$ is the function which takes a real number x as argument and outputs the greatest integer smaller or equal to x .

Hint 1: Try with small values of n .

Hint 2: Use case distinction and the pigeonhole principle.

Exercise 4.2 Element or Subset (★)

For each of the following choices of sets A and B , decide which of the statements $A \in B$ and $A \subseteq B$ are true.

1. $A = \{1, 0, \{0\}, 1\}$, $B = \{\{0, 1, 0, \{0, 0\}\}, 1, 10, 0\}$
2. $A = \emptyset$, $B = \{\{\emptyset\}, \{\emptyset, \emptyset\}, \emptyset\}$
3. $A = \{\{0\}, 0, \{0\}, \{\{0\}\}\}$, $B = \{0, \emptyset, \{\{0\}\}, \{0\}\}$
4. $A = \{\emptyset\}$, $B = \{\emptyset, \{\emptyset, \emptyset, \emptyset\}\}$

Exercise 4.3 Operations on Sets (★)

In each of the following cases, give a set A such that

1. There exists an $x \in A$ such that $x \subseteq A$.
2. $A \not\subseteq \mathcal{P}(A)$ and there exists an $x \in A$ such that $x \subseteq \mathcal{P}(A)$.
3. $A \subseteq \mathcal{P}(A)$ and for all $x \in A$ it holds that $x \not\subseteq \mathcal{P}(A)$.

Exercise 4.4 Cardinality (★)

Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$ and $B = \{A, \{\emptyset\}, \{\{\emptyset\}\}$. Specify each of the following sets (by listing all its elements) and give its cardinality.

1. $A \cup B$
2. $A \cap B$
3. $\emptyset \times A$
4. $\{0\} \times \{3, 1\}$
5. $\{\{1, 2\}\} \times \{3\}$
6. $\mathcal{P}(\{\emptyset\})$

Exercise 4.5 Relating Two Power Sets (★ ★)

Prove or disprove each of the following statements.

1. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$ for any sets A and B .
2. $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$ for any sets A and B .
3. $A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$ for any sets A and B .

Exercise 4.6 Interview Exercise — GRADED

(6 points)

This exercise will be discussed during the interviews in the week of the 12/10/2026. You are allowed to bring *handwritten* notes to the interview.

1. (2 points) (★)

Let us recall some definitions about set theory. In the following consider the sets $A = \{1, 2\}$, $B = \{3, a\}$, $C = \{1, 2, 3, a\}$.

- (a) What does it mean for a set S to be a *subset* of a set T ? Which of the above sets A , B and C are subsets of each other?
- (b) What is the *power set* of a set S ? What is the power set of A ?

2. (1 point) (★)

Consider the set $A = \{1, 2, \emptyset\}$. List all elements of the set $S = \{A \cup \{\emptyset\}, A \cap \{\emptyset\}\} \times \{A \times \emptyset\}$ as well as of $\mathcal{P}(S)$. Your listed elements should not contain the operators $\cup$, $\cap$ or $\times$. You can write A or $\{1, 2, \emptyset\}$ according to your preference. Briefly explain how you arrived at your solution.

3. (3 points) (★ ★)

Prove that for any sets A, B, C it holds that

$$(B \setminus C) \cup (A \setminus B) = (B \cup A) \setminus (C \cap B).$$

Your prove should not use any of the laws given in Theorem 3.4 of the lecture notes.

Expectation. Your proof should be a sequence of logical equivalences. Each step should consist of applying the definition of $\cup, \cap, \setminus, \subseteq$ or $\emptyset$ respectively, one of the logic rules given in Lemma 2.1 of the lecture notes¹, or one of the following rules: $(F \vee G) \wedge ((H \vee G) \wedge (H \vee \neg F)) \equiv (F \vee G) \wedge (H \vee \neg F)$ (consensus theorem), $F \rightarrow G \equiv \neg F \vee G$, $F \wedge \neg F \equiv \perp$, $F \wedge \perp \equiv \perp$, $F \vee \perp \equiv F$, $F \vee \neg F \equiv \top$, $F \wedge \top \equiv F$, and $F \vee \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1 3). Each step of your proof should apply only a **single** rule and state **which** rule was applied. In this exercise, you may **apply the same rule multiple times** within a single proof step. Remember that applying commutativity requires a proof step.

Exercise 4.7 Special Families of Sets (★ ★)

Let $X \neq \emptyset$ be a set. We define the following predicate:²

$$Q_X(\mathcal{A}) = 1 \iff \begin{cases} \mathcal{A} \subseteq \mathcal{P}(X), \\ \mathcal{A} \neq \emptyset, \\ A \cup B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ A \cap B \in \mathcal{A} \text{ for all } A, B \in \mathcal{A}, \\ X \setminus A \in \mathcal{A} \text{ for all } A \in \mathcal{A}. \end{cases}$$

Prove or disprove the following statements.

1. $Q_X(\mathcal{P}(X)) = 1$.
2. $Q_X(\{X\}) = 1$.
3. For all $\mathcal{A} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ then $X \in \mathcal{A}$.
4. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cup \mathcal{B}) = 1$.
5. For all $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$, if $Q_X(\mathcal{A}) = 1$ and $Q_X(\mathcal{B}) = 1$ then $Q_X(\mathcal{A} \cap \mathcal{B}) = 1$.

Exercise 4.8 Short questions (exam 2022) (★)

Just write down the correct answer. No justification is required.

¹Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).

²This notation stand for the logical *conjunction* of all statements on the right, meaning the predicate is true if and only if *all* statements on the right are true.

1. How many elements does the set $\{\{0, 1\}, \{0\} \times \{1\}\} \times \{0\}$ have?
2. List each element of the set $\{\emptyset, (0, 1)\} \times \{\{0\} \cup \{1\}, \{1, 0\}\}$ exactly once.
3. Find sets A , B , and C such that $A \setminus B \subset A \setminus C$.
4. Find a set A such that $A \cap \mathcal{P}(A) \neq \emptyset$.

Exercise 4.9 Short questions (exam 2021) (★)

Just write down the correct answer. No justification is required.

1. True or False: the sets $\emptyset \times \emptyset$ and $\{\emptyset\} \times \{\emptyset\}$ are equal.
2. True or False: the set $\{\emptyset, \{\emptyset\}\}$ is a subset of $\{\emptyset\}$.
3. True or False: for any two finite sets A and B we have $|A \cup B| = |A| + |B|$.
4. True or False: for any two sets A and B there exists a set C such that

$$A = B \cap C \quad \text{or} \quad B = A \cap C.$$

5. True or False: for any set A , we have $A \cap \mathcal{P}(A) = \emptyset$.