

Diskrete Mathematik HS2026 — Prof. Dennis HOFHEINZ

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Exercise sheet 3

This is the exercise sheet number 3. The difficulty of the questions and exercises are rated from very easy (★) to hard (★★★★). The graded exercise is Exercise 3.2. If your interview is in the week of the **05/10/2026**, please prepare this exercise for the interview. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 3.1 Expressing Relationship of Humans in Predicate Logic (★)

Consider, as in the lecture, the universe of all humans (including those who died) and the following predicate:

$$\text{par}(x, y) = 1 \iff "x \text{ is parent of } y."$$

Express the following statements as a formula in predicate logic, using only the above predicates (in particular, do **not** use the predicate equals, often also written as =).

1. x is great-grandparent of y .
2. x and y are (first) cousins.

Exercise 3.2 Squirrels and Horses — GRADED

(6 points)

*This exercise will be discussed during the interviews in the week of the 05/10/2026. You are allowed to bring **handwritten** notes to the interview.*

1. (2 points) (★)

Let us recall some basics about predicate logic.

- (a) (1 point) Consider the following formula:

$$F = \forall x \exists y (P(x) \vee Q(x, f(y))).$$

In order to be able to reason about predicate logic formulas, we need to define an interpretation. In the example above, what would this interpretation need to define?

- (b) (1 point) Compare the following two formulas. What are the differences?

$$P(x) \quad \text{and} \quad \forall x P(x).$$

2. (2 points) (★)

Consider the universe of all animals. Apparently, horses and squirrels are particularly competitive. They have decided to organize races in order to determine who is the fastest. Thus, we define the following predicates.

$$\text{sq}(x) = 1 \iff "x \text{ is a squirrel}."$$

$$\text{hs}(x) = 1 \iff "x \text{ is a horse}."$$

$$\text{faster}(x, y) = 1 \iff "x \text{ is faster than } y."$$

- (a) (1 point) Express the following statement in natural language as a formula in predicate logic, using only the predicates above: Horses are faster than squirrels. (Every given horse is faster than every squirrel).
- (b) (1 point) Express the following formula in predicate logic in natural language.

$$\exists x \forall y (\text{faster}(y, x) \rightarrow (\forall z (\text{faster}(y, z))))).$$

3. (2 points) (★ ★)

Find a universe U and define the predicates such that one of the following formulas is true but not the other. Try to come up with a very simple and compact solution, so you have enough time to explain and justify it.

(a)

$$\forall x (P(x) \rightarrow (\exists y \forall z (Q(z, y) \vee R(x, z))))$$

(b)

$$\exists y \forall x (P(x) \rightarrow ((\forall z Q(z, y) \vee (\forall z R(x, z)))).$$

Exercise 3.3 Winning Strategy (★ ★)

Alice and Bob play a game in which the stake is a chocolate bar. Rules of the game are the following: Alice chooses two integers a_1, a_2 and Bob chooses two integers b_1, b_2 . Alice wins whenever $a_1 + (a_2 + b_1)^{|b_2|+1} = 1$ and Bob wins otherwise.

1. First, consider the case when Alice and Bob announce all their numbers at the same time. Give a formula that describes the statement "Alice has a winning strategy." Is the statement expressed by the formula true?
2. In the second case, Alice and Bob announce their numbers one by one. That is, first Alice announces a_1 , then Bob announces b_1 , then Alice announces a_2 , and at the end Bob replies with b_2 . Once again, give a formula that describes the statement "Alice has a winning strategy." Is the statement expressed by the formula true in this case?

Exercise 3.4 Indirect Proof of an Implication

Prove indirectly that for all natural numbers $n > 0$, we have:

1. (★) If n^2 is odd, then n is also odd.
2. (★ ★) If $42^n - 1$ is a prime, then n is odd.

Exercise 3.5 Case Distinction

Prove by case distinction that:

1. (★) $n^3 + 2n + 6$ is divisible by 3 for all natural numbers $n \geq 0$.
2. (★ ★) If p and $p^2 + 2$ are primes, then $p^3 + 2$ is also a prime.

Exercise 3.6 Proof by Contradiction

1. (★ ★) Show by contradiction that the sum of a rational number and an irrational number is irrational.

Hint: Use the fact that the difference of two rational numbers is rational.

2. (★ ★ ★) Show that the number $2^{\frac{1}{n}}$ is irrational for $n > 2$, by reaching a contradiction with Fermat's Last Theorem.

Hint: Fermat's Last Theorem states that no positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for $n > 2$.

Exercise 3.7 New Proof Patterns (★)

For each of the following proof patterns, **prove** or **disprove** that it is sound. Do so by first writing it as a statement involving logical consequence on formulas and then proving that the resulting statement is either true or false.

1. (★) To prove a statement S , find two appropriate statements T_1 and T_2 . Assume that S is false and show (from this assumption) that at least one of the statements T_1 and T_2 is true. Then show that at least one of the statements T_1 and T_2 is false.
2. (★) To prove an implication $S \Rightarrow T$, find an appropriate statement R . First, show that R is false. Then, assume that S is true and T is false, and prove that (from these assumptions) R is true.

Exercise 3.8 Short Questions (Old Exam Questions) (★ ★)

This exercise contains several short questions from exams from previous years.

1. Find a formula F in predicate logic, such that (F is satisfiable and) in every interpretation for F such that F is true, the universe has at least 2 elements. You may not use equality ($=$) in your solution.
2. Find a formula F which contains a binary predicate symbol P such that
 - for $U = \mathbb{N}$ and $P = \leq$, F is true
 - for $U = \mathbb{N} \setminus \{0\}$ and $P = |$, F is not true. (Here $|$ denotes the divisibility relation, e.g. $x | y = 1 \iff x$ divides y .)
3. Let $F = \forall z \left((\exists x P(x)) \vee Q(x, z) \right)$ and let

$$U = \mathbb{N}, x = 29, P(x) = 1 \iff 2 | (x^2 + x + 1), Q(x, y) = 1 \iff x + y > 5.$$

Is F true with this interpretation? ¹

Due by 05/10/2026, 23:59.
Exercise 3.2 will be graded.

¹The exam asked to prove or disprove this claim. We will see how to formally prove and disprove such claims in Chapter 6. For now, an informal justification suffices.