

Diskrete Mathematik HS2026 — Prof. Dennis HOFHEINZ

Milan GONZALEZ-THAUVIN — Zoé REINKE — Georgette WEINGÄRTNER

Exercise sheet 2

This is the exercise sheet number 2. The difficulty of the questions and exercises are rated from very easy (★) to hard (★★★★). The graded exercise is Exercise 2.3. If your interview is in the week of the **28/09/2026**, please prepare this exercise for the interview. The solution to this exercise must be your own work, you may not share your solutions with anyone else. See also the note on dishonest behavior on the Moodle page.

Exercise 2.1 Logical Consequence (★)

Prove or disprove the following statements about formulas.

1. $A \wedge (A \rightarrow B) \models B$
2. $A \rightarrow B \models \neg A \rightarrow \neg B$
3. $\models (A \rightarrow B) \vee (B \rightarrow A)$
4. $(A \rightarrow B) \wedge (B \rightarrow C) \models (A \rightarrow C)$

Exercise 2.2 Satisfiability and Tautologies (★)

For each of the formulas below, determine whether it is satisfiable or unsatisfiable and whether it is a tautology or not. Prove your answers.

1. $(A \vee B) \wedge \neg A$
2. $((A \rightarrow B) \wedge (B \rightarrow C)) \wedge \neg(A \rightarrow C)$

Exercise 2.3 Interview Exercise — GRADED

(6 points)

*This exercise will be discussed during the interviews in the week of the 28/09/2026. You are allowed to bring **handwritten** notes to the interview.*

1. (2 points) (★)

Let us recall some definitions about propositional logic.

- (a) What does it mean for two formulas F and G in propositional logic to be *equivalent*?
- (b) Let F be a formula of propositional logic. What does $\models F$ mean? If $\models F$ holds, then what can we say about $\neg F$?

2. (1 point) (★)

Lemma 2.1. of the lecture notes lists many basic equivalences of propositional logic. Show that $A \wedge (A \vee B) \equiv A$ by comparing function tables.

3. (3 points) (★★)

Prove that the formula $F = ((A \rightarrow B) \rightarrow A) \rightarrow A$ is a tautology, i.e. that $\models F$, using a sequence of **at most 9** equivalence transformations.

Expectation. Your proof should be in the form of a sequence of **at most 9** steps, where each step consists of applying the rule $F \rightarrow G \equiv \neg F \vee G$, one of the rules given in Lemma 2.1 of the lecture notes¹, or one of the following rules: $F \wedge \neg F \equiv \perp$, $F \wedge \perp \equiv \perp$, $F \vee \perp \equiv F$, $F \vee \neg F \equiv \top$, $F \wedge \top \equiv F$, and $F \vee \top \equiv \top$. For this exercise, associativity is to be applied as in Lemma 2.1 3). Each step of your proof should apply only a **single** rule and state **which** rule was applied. You may not apply the same rule multiple times within a single proof step. Remember that applying commutativity requires a proof step.

Exercise 2.4 Knights and Knaves (★★★)

We find ourselves on a strange island with only two types of inhabitants: knights and knaves. The knights always tell the truth, while the knaves always lie. From the outside, both groups look exactly the same and we cannot distinguish one from the other.

We have lost our way and come to a fork in the road. We know that one of the roads leads to a deadly jungle, while the other will take us to a friendly village. We see an islander standing at the fork. He is willing to answer only one question and his answer can only be “True” or “False”. What question do we ask?

We want to use propositional logic to solve this problem. Let A be the proposition “The left road leads to the village.” and let B be the proposition “The islander is a knight.” We phrase our question as a formula F in A and B , asking the islander about the truth value of F . How do we choose F such that we are guaranteed to learn which road leads to the village?

Exercise 2.5 Quantifiers and Predicates

In this exercise the universe is fixed to the set $\mathbb{Z}$ of integers.

1. For each of the following statements, write a formula, in which the only predicates are `less`, `equals` and `prime` (instead of `less( $n, m$ )` and `equals( $n, m$ )` you can write $n < m$ and $n = m$ accordingly).² You can also use the symbols $+$ and $\cdot$ to denote addition and multiplication. Which of them are true? (You don’t need to justify this.)

¹Lemma 2.1 states rules involving propositional symbols, but you may apply those rules at the level of formulas (see Section 2.3.5 of the lecture notes).

²Note that `less( $n, m$ )` is true if n is **strictly** smaller than m , so it is false for $n = m$.

- (a) (★) If the product of two integer numbers is positive, then at least one of these numbers is positive.
- (b) (★) For every natural number, one can find a strictly greater natural number that is divisible by 3.
- (c) (★ ★) Every even integer greater than 2 is a sum of two primes.

2. Consider the following predicates $P(x)$ and $Q(x, y)$:

$$P(x) = \begin{cases} 1, & x > 0 \\ 0, & \text{otherwise} \end{cases} \quad Q(x, y) = \begin{cases} 1, & xy = 1 \\ 0, & \text{otherwise} \end{cases}$$

In this context, describe the following statements in words. Also, for each statement, decide whether it is true or false.

- (a) (★) $\forall x \exists y Q(x, y)$
- (b) (★) $\exists x (\forall y \neg Q(x, y) \wedge \exists y P(y))$

Exercise 2.6 Short Questions (Old Exam Questions)

This exercise contains several short questions from exams from previous years.

1. Find two propositional formulas F and G , such that $F \rightarrow G$ is unsatisfiable.
2. The statement " $F \wedge G$ is satisfiable if and only if F is satisfiable and G is satisfiable" is true for any formulas F and G . True or False?
3. A formula F in propositional logic is unsatisfiable if and only if $\neg F$ is a tautology. True or False?
4. The following formula is a tautology: $A \rightarrow (\neg A \rightarrow B)$. True or False?